Self-replicating pulses and Sierpinski gaskets in excitable media

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In our previous papers, we have shown by computer simulations that a Sierpinski gasket pattern appears in a Bonhoeffer–van der Pol type reaction-diffusion system. In this paper, we show another class of regular self-similar structure which is found in four different excitable reaction-diffusion systems. This result strongly implies that the existence of the self-similar spatiotemporal evolution is universal in excitable reaction-diffusion media.

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I. INTRODUCTION

Recently, a rich variety of pulse dynamics have been found in nonlinear open systems. First of all, computer simulations of several reaction-diffusion systems have revealed that a propagating pulse is stable upon collision with another pulse, that is, a pair of counterpropagating pulses undergoes an elasticlike collision [1–4]. It is also possible that a pulse pair is deformed during collision but survives again just like a soliton in an integrable nondissipative system [5,6]. It is noted that these unexpected behaviors in nonlinear dissipative systems occur only in a limited region of the parameters. For most of the parameter region, pulses annihilate upon collision as is usual in a dissipative system.

Another interesting property of pulses is self-replication, which has been observed not only in computer simulations but also in a real experiment [7]. In the Gray-Scott model, a motionless pulse splits into two pulses which grow and repeat self-replication until the density of pulses is sufficiently large [1,8]. A propagating pulse can also self-replicate in which a daughter pulse is emitted, known as a backfiring phenomenon [1,9].

It is important to note that these three basic properties of pulses—pair annihilation, preservation upon collision, and self-replication—can coexist in a small but finite parameter region. In this situation, we have shown in previous papers [6,10] that the interplay among the three properties causes a regular self-similar spatiotemporal evolution of a trajectory of pulses as shown in Fig. 1. Preservation is possible only for completely symmetric collision. As a result, the same pulse trajectory is generated every three generations. This is isomorphic to a Sierpinski gasket (SG) generated by a cellular automation,

$$a^{t+1}(i)=a^{t}(i-1)+a^{t}(i+1), \quad \text{mod } k, \tag{1}$$

where $a^{t}(i)=0,1,\ldots,k-1$ is defined on a one-dimensional lattice. The pattern in Fig. 1 corresponds to the case $k=3$.

It is mentioned here that Meinhardt has also been concerned with a connection between reaction-diffusion systems and cellular automata in his study of biological pattern formation such as sea shells [11] where, however, no perfect fractal structure has been obtained.

The purpose of this paper is to explore how generic the formation of a regular self-similar pattern is. We will show that a pattern like Fig. 1 is not an exceptional one for a particular set of reaction-diffusion systems. In four different model systems, a self-similar pattern equivalent to Eq. (1) with $k=2$ will be obtained. The results definitely suggest that the existence of a self-similar spatiotemporal evolution is universal in excitable reaction-diffusion media.

This paper is organized as follows. In Sec. II, we introduce the Bonhoeffer–van der Pol (BvP) type reaction-diffusion system with a cubic nonlinear term. It is shown that both the traveling pulse and the breathing pulse undergo self-replication, which causes generally complex spatiotemporal patterns. However, we show that, if one tunes the parameters, SG with $k=2$ emerges.

In Sec. III, we investigate the BvP model with a hyperbolic tangent nonlinear term. An SG with $k=3$ in Fig. 1 was generated by this model [6,10]. We show, however, that for different values of the parameters an SG with $k=2$ is also realized in this model.

In Sec. IV, we carry out numerical simulations of the Gray-Scott model and obtain an SG with $k=2$. In Sec. V, we investigate the Prague model, which was introduced to study self-replicating waves observed in the Belousov-Zhabotinsky

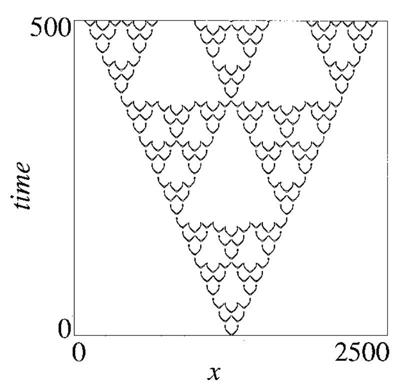

FIG. 1. Spatiotemporal pattern of interacting pulses in the BvP model equations with a hyperbolic nonlinear term where $\alpha=0.1$, $\gamma=0$, $\delta=0.05$, $\tau=0.34$, $I=0$, $D_v=10$, and $D_u=1$. The lines indicate the contour line of $u=0.2$. The quantities in this and all other figures below are dimensionless.

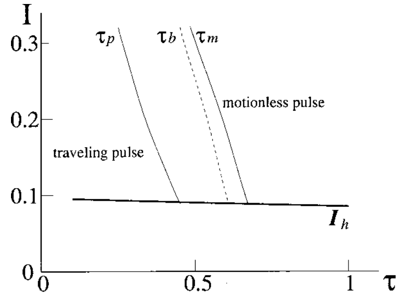

FIG. 2. The phase diagram in the I-τ plane.

reaction [12]. Numerical simulations of this model show that an SG with $k=2$ exists in a small parameter region. In all of the simulations in Sec. II–V, the Neumann boundary conditions are used at the system boundaries. Finally, in Sec. VI, we give a discussion of our results.

II. BvP MODEL WITH A CUBIC NONLINEARITY

A. Model equation

In our previous papers [6,10], we have reported that a regular self-similar spatiotemporal pattern like an SG appears in Bonhoeffer–van der Pol type reaction-diffusion equations,

$$\tau \frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + f(u) - v, \quad (2)$$

$$\frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} + u - \gamma v + I, \quad (3)$$

where $D_u > 0$ and $D_v > 0$ are the diffusion rates of u and v, respectively. The parameters τ, γ, and I are assumed to be positive. In this section, we shall explore the pulse dynamics of the BvP model (2) and (3) with the cubic nonlinearity

$$f(u) = au(u+1)(1-u), \quad (4)$$

where a is a positive constant. The parameters are chosen such that the system Eqs. (2) and (3) with Eq. (4) is excitable. Throughout this section, we set $D_u = 1$, $D_v = 10$, $a = 5$, and $\gamma = 0.25$, and examine the behavior of pulses by changing the values of τ and I.

In the spatially homogeneous case, the set of Eqs. (2) and (3) with Eq. (4) has a subcritical Hopf bifurcation point $I \equiv I_h$ such that a limit cycle solution appears when $I < I_h$. Our concern is the case $I > I_h$, where the system is excitable.

Equations (2) and (3) with Eq. (4) have been studied in detail theoretically [13–15]. As shown in Fig. 2, there are three bifurcation points by changing the parameters τ. A traveling pulse is stable for $\tau < \tau_p$, while a motionless pulse is stable for $\tau > \tau_m$. In the present choice of the parameters, τ_p is always smaller than τ_m. The dependence of τ_p and τ_m on I is schematically shown in Fig. 2. There is another special value τ_b such that a motionless pulse loses stability and undergoes a breathing motion in the interval $\tau_b < \tau < \tau_m$.

It should be noted that there is an interval $\tau_p < \tau < \tau_b$ where neither the breathing pulse nor the traveling pulse exists as a stationary state. This is the very region where self-replication of pulses is observed and hence rich varieties of spatiotemporal patterns appear as described below.

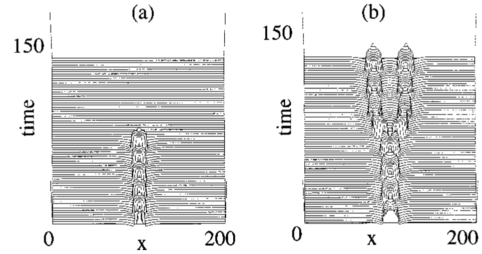

FIG. 3. (a) Annihilation of an oscillating pulse for $I=0.2, \tau = 0.53$ ($\tau_b = 0.54$). (b) Self-replication of an oscillating pulse for $I=0.1, \tau=0.59$ ($\tau_b=0.60$). The lines indicate the profile of u.

B. Self-replication of pulses

We have carried out numerical simulations of Eqs. (2) and (3) with Eq. (4) in the region $\tau_p < \tau < \tau_b$ shown in Fig. 2. The time mesh is 0.001 while the space mesh is 0.25.

We provide a stable breathing pulse at $\tau \gtrsim \tau_b$ and then change the parameter τ to $\tau \lesssim \tau_b$. The result for $I=0.2$ and $\tau = 0.53$ is shown in Fig. 3(a), where one can see that a pulse breathes for a finite time and vanishes by collision of two interfaces of the pulse.

However, when I is close to the Hopf bifurcation threshold I_h, a breathing pulse splits into two pulses. An example is shown in Fig. 3(b) for $I=0.1$ and $\tau=0.59$. It is evident that when the width of the breathing pulse becomes maximum, the value of u in the middle of the pulse decreases rapidly so that replication occurs.

The long-time behavior of simulations of Fig. 3(b) is shown in Figs. 4(a) and 4(b), which display spatiotemporal evolutions of pulses. In Fig. 4(a) for $\tau = 0.59$, breathing pulses disappear after a few times of self-replicating. When $\tau = 0.58$, Fig. 4(b) shows that breathing pulses repeat the self-replicating process without extinction so that the number of pulses increases in time.

A traveling pulse for $\tau \gtrsim \tau_p$ behaves similarly. We provide a pulse for $\tau \lesssim \tau_p$. This pulse propagates for some finite interval after an abrupt increase of τ larger than τ_p. During propagation, the pulse changes its shape to a symmetric form and then either vanishes or self-replicates depending on the parameters. When I is much larger than I_h, the pulse van-

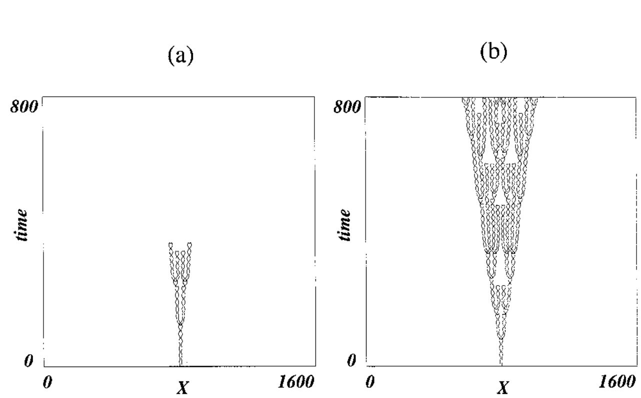

FIG. 4. Spatiotemporal pattern for (a) $\tau = 0.59$ and (b) $\tau = 0.58$. Other parameters are the same as those in Fig. 3(b). The lines indicate the contour line of $u=0$.

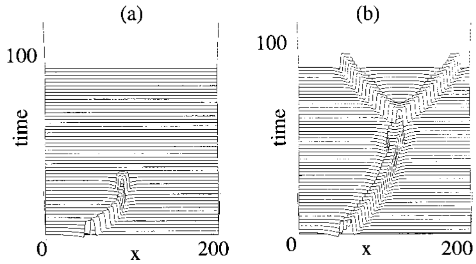

FIG. 5. Self-replication of a traveling pulse for (a) $I=0.3,\tau=0.28$ ($\tau_p=0.27$) and (b) $I=0.1,\tau=0.44$ ($\tau_p=0.43$). The lines indicate the contour line of $u=0$.

ishes as shown in Fig. 5(a) for $I=0.3$ and $\tau=0.28$. When I is close to I_h, self-replication occurs as in Fig. 5(b) for $I=0.1$ and $\tau=0.44$.

Figure 6(a) shows a spatiotemporal pattern of self-replicating traveling pulses for the same parameters as in Fig. 5(b). The initial condition is a motionless pulse generated for $\tau=1$. It can be seen that self-replication of traveling pulses occurs irregularly. The self-replicating pulses annihilate or reflect upon collision so that an apparently chaotic spatiotemporal pattern appears.

C. Regular self-similar patterns

When the value of τ is in the middle of the region $\tau_p<\tau<\tau_b$ for $I\gtrsim I_h$, we obtain a regular self-similar pattern as shown in Fig. 6(b). Here, we have used the same initial condition as in Fig. 6(a).

Note that the SG in Fig. 6(b) differs from Fig. 1. The most crucial property is that preservation of pulses does not exist in Fig. 6(b). All of the pulses undergo pair annihilation upon collision so that the same trajectory of the pulses appears every two generations. The SG in Fig. 6(b) is equivalent with Eq. (1) with $k=2$.

The SG in Fig. 6(b) is robust in comparison with other irregular spatiotemporal patterns. For example, the pattern in Fig. 6(a) is not reproduced when we add a noise to the initial conditions and thus it is sensitive to the initial conditions.

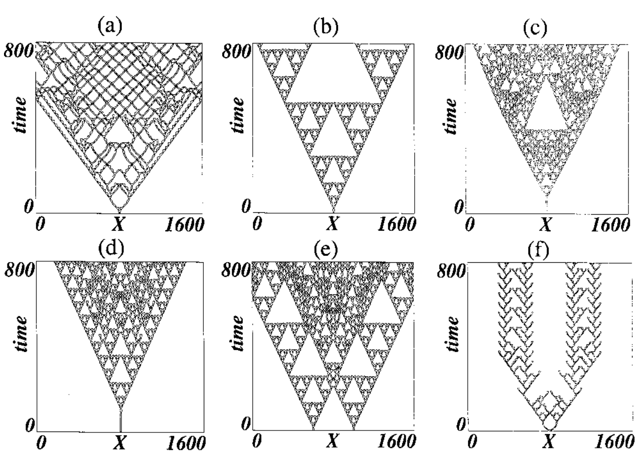

FIG. 6. Spatiotemporal pattern for (a) $\tau=0.44$, (b) $\tau=0.50$, (c) $\tau=0.48$, and (d) $\tau=0.51$. (e) Collision of two SG's starting with two pulses at $x=600$ and 1000. The parameters are the same as in Fig. (b). (f) Periodic pattern for $\tau=0.34$, $I=0.15$, $a=4$, $\gamma=0.3$, $D_u=1$, and $D_v=10$. The lines indicate the contour line of $u=0$.

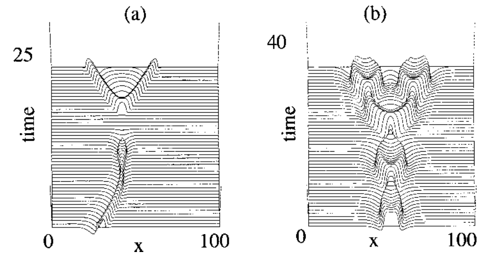

FIG. 7. Self-replication of a pulse in the BvP model with a hyperbolic nonlinear term. The parameters are (a) $\alpha=0.1$, $\gamma=0$, $\delta=0.05$, $\tau=0.34$, $I=0$, $D_v=10$, and $D_u=1$; (b) $\alpha=0.105$, $\gamma=0.21$, $\delta=0.05$, $\tau=0.4$, $I=0$, $D_v=10.5$, and $D_u=1$.

However, we have verified that the SG in Fig. 6(b) survives even for initial conditions deformed by noise. Furthermore, an asymmetric initial profile like a traveling pulse also produces the same SG. It seems that any localized pulse whose magnitude is larger than a certain threshold produces the SG.

The SG in Fig. 6(b) exists in the parameter region $0.49\leqslant\tau\leqslant0.50$ with the other parameters fixed as in Fig. 6(a). Outside of this region, the self-similarity does not last indefinitely as shown in Fig. 6(c) for $\tau=0.48$ and in Fig. 6(d) for $\tau=0.51$.

We have also examined collision of two SG domains. Figure 6(e) displays the result where we start with the same parameters as in Fig. 6(b). The SG is generally destroyed by the collision causing complicated evolution of pulses. Only when two pulses are at a particular distance initially is the SG preserved after collision.

We make a final remark that a regular pattern also emerges in the interval $\tau_p<\tau<\tau_b$ although it is not self-similar. One example is shown in Fig. 6(f) for $a=4$ and $\gamma=0.3$ and for the initial condition $u(x,0)=2\exp(-x^2)+u_0$ and $v(x,0)=v_0$ with the equilibrium uniform values u_0 and v_0. After an initial transient, the system enters a cycle in which a branched pattern is generated repeatedly.

III. BvP MODEL WITH A HYPERBOLIC TANGENT NONLINEARITY

In the preceding section, we have obtained an SG with $k=2$ in the excitable system (2) and (3) with Eq. (4). This is quite in contrast to our previous result that an SG with $k=3$ was obtained in Eqs. (2) and (3) with a hyperbolic tangent nonlinearity,

$$f(u)=\frac{1}{2}\left(\tanh\frac{u-\alpha}{\delta}+\tanh\frac{\alpha}{\delta}\right)-u, \tag{5}$$

where α and δ are positive constants. The essential difference is that Eqs. (2) and (3) with Eq. (5) are bistable in the sense that a stable uniform solution and a stable limit cycle solution coexist [6,10,16].

In this section, we will show that an SG with $k=2$ can be realized even for the hyperbolic tangent nonlinearity. When $\gamma=0$ in Eq. (3), the system is bistable in the above sense and a pulse self-replicates as shown in Fig. 7(a), where a pulse becomes small for a short period [17] and survives again splitting into two pulses. The SG with $k=3$ in Fig. 1 was obtained in this situation [6]. On the other hand, when γ

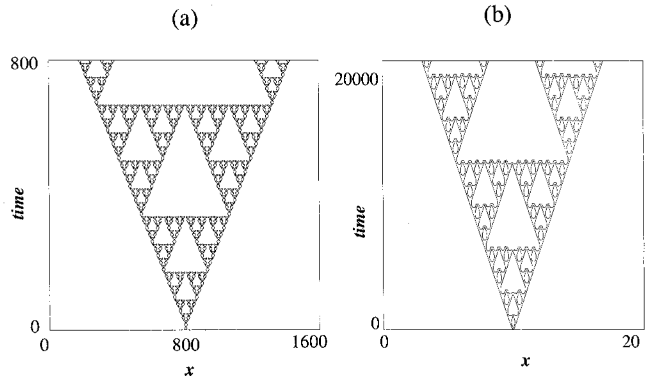

FIG. 8. (a) SG with $k=2$ for the BvP model with a hyperbolic nonlinear term. The parameters are $\alpha=0.105$, $\gamma=0.21$, $\delta=0.05$, $\tau=0.4$, $D_v=10.5$, and $D_u=1$. The lines indicate the contour line of $u=0$. (b) SG with $k=2$ for the Gray-Scott model. The parameters are $F=0.0253$, $k=0.0525$, $D_u=1.15\times10^{-5}$, and $D_v=1.0\times10^{-5}$. The lines indicate the contour line of $u=0.5$.

$=0.21$ and $\tau=0.4$, a pulse self-replicates as in Fig. 7(b). It should be noted that this behavior of self-replication is qualitatively different from that in Fig. 7(a) but quite similar to those in Figs. 3(b) or 5(b).

We have carried out numerical simulations of Eqs. (2) and (3) with Eq. (5) in the parameter region where the self-replication occurs as in Fig. 7(b). Other parameters are chosen to be almost the same as the SG with $k=3$. An SG with $k=2$ is really obtained as shown in Fig. 8(a). The time mesh is 0.001 and the space mesh is 0.125 for these computations.

Thus, we have obtained two SG's with $k=2$ as in Figs. 6(b) and 8(a). However, it is remarked that there is a small difference in these figures. By looking at the contour lines of u, one notes that a pulse in Fig. 6(b) splits into two during the first breathing oscillation whereas a pulse in Fig. 8(a) self-replicates during the second oscillation.

IV. GRAY-SCOTT MODEL

In Secs. II and III, we have shown that SG's with $k=2$ appear in the BvP model having two different nonlinear terms. Here, it will be shown that an SG emerges in the Gray-Scott model given by the following set of equations:

$$\frac{\partial u}{\partial t}=D_u\frac{\partial^2 u}{\partial x^2}-uv^2+F(1-u), \tag{6}$$

$$\frac{\partial v}{\partial t}=D_v\frac{\partial^2 v}{\partial x^2}+uv^2-(F+k)v, \tag{7}$$

where $D_u>0$ and $D_v>0$ are the diffusion coefficients. We assume that F and k are positive constants.

Self-replication of a pulse in the Gray-Scott model has been studied both numerically and analytically [1,8,18,19]. However, the spatiotemporal evolution of the interacting pulses has not attracted much attention because almost all of the previous studies focused on the parameters such that there is a repulsive interaction between pulses. Hence, the system approaches a time-independent state asymptotically.

It has been found, however, that there is a small region of the parameters where pulses annihilate upon collision and a chaotic pattern emerges [20]. We carry out simulations of Eqs. (6) and (7) in this parameter region but with a slightly different value of D_u and D_v. Actually, in Ref. [20], these are set to be $D_u=2\times10^{-5}$ and $D_v=10^{-5}$. We could not find any SG for these values of the diffusion constants. In the present study, we chose $D_u=1.15\times10^{-5}$ and $D_v=10^{-5}$. As shown in Fig. 8(b), we obtain an SG with $k=2$ for $F=0.0253$ and $k=0.0525$. The same SG is obtained for $D_u=1.16\times10^{-5}$ without changing the other parameters. The initial condition is such that $u=0.5$ and $v=1$ just at the center of the system and $u=1$ and $v=0$ otherwise. The time mesh is 0.1 and the space mesh is 0.01.

The SG shown in Fig. 8(b) corresponds to a cellular automaton (1) with $k=2$. A question arises whether an SG with $k=3$ is possible in the Gray-Scott model or not as in the BvP model with a hyperbolic tangent nonlinearity. A necessary condition for an SG with $k=3$ is the coexistence of pair annihilation, self-replication, and preservation of pulses [10]. We have confirmed that there is a parameter region in the Gray-Scott model where these basic properties coexist. However, we have not succeeded in realizing an SG with $k=3$ in the Gray-Scott model. This is due to the fact that the interval during a collision (phase shift) is substantially longer than that of self-replication. This unbalance of the two time intervals makes the formation of SG with $k=3$ impossible. On the other hand, in the BvP equation with a hyperbolic nonlinearity, these two time intervals are almost the same so that a self-replication repeats coherently with preservation at each generation.

V. PRAGUE MODEL

The fourth example of an excitable system where an SG appears is the following two-component reaction-diffusion model:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{1}{\epsilon}u[c-u+(1-c)v]\left(u-\frac{v+b}{a}\right), \tag{8}$$

$$\frac{\partial v}{\partial t}=D\frac{\partial^2 v}{\partial x^2}+u-v, \tag{9}$$

where $D>0$ is the diffusion coefficient and a, b, c, and ϵ are positive constants. We call this model the Prague model.

Marek *et al.* [12] have introduced Eqs. (8) and (9) to study numerically the splitting of a reduction wave in the Belousov-Zhabotinsky reaction. They have found that self-replication of pulses occurs for $a=0.99$, $b=0.01$, $c=0.2$, $D=1$, and $\epsilon=0.01$. Here, we investigate the behavior of the Prague model by changing the values of D and ϵ. An SG with $k=2$ appears for $D=1.2$ and $\epsilon=0.009$ as shown in Fig. 9(a). The initial condition is $u(x,0)=\exp(-x^2)$ and $v(x,0)=0$. The time mesh is 0.001 and the space mesh is 0.25.

When the diffusion constant D in Eq. (9) is decreased, one has an entirely different pulse dynamics. Pulses do not undergo pair annihilation upon collision. For instance, when $D=0.5$ (and $\epsilon=0.01$), we have the spatiotemporal evolution in Fig. 9(b) where preservation of pulses is repeated so that a regular periodic pattern is formed. This pattern corresponds to the cellular automaton (1) with $k=\infty$.

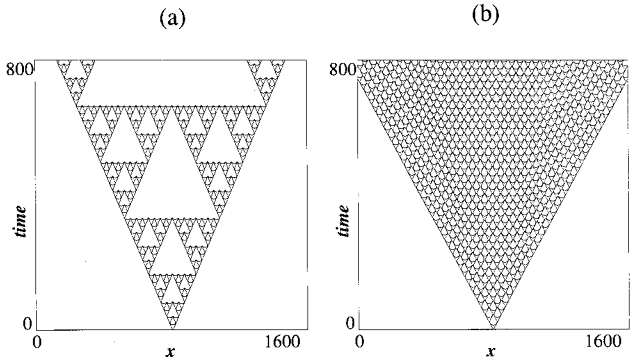

FIG. 9. (a) SG with $k=2$ for the Prague model. The parameters are $a=0.99$, $b=0.01$, $c=0.2$, $D=1.2$, and $\epsilon=0.009$. (b) Spatiotemporal pattern for the Prague model. The parameters are $a=0.99$, $b=0.01$, $c=0.2$, $D=0.5$, and $\epsilon=0.01$. The lines indicate the contour line of $u=0.2$.

VI. DISCUSSION

In this paper, we have shown by numerical simulations that a Sierpinski gasket pattern with $k=2$ can be produced in four different excitable reaction-diffusion systems: (i) the BvP model with a cubic nonlinear term, (ii) the BvP model with a hyperbolic tangent nonlinear term, (iii) the Gray-Scott model, and (iv) the Prague model. We expect from these results that the SG is very common to excitable media [21].

It is definite that an SG cannot be realized without self-replication of pulses. So far, there have been several studies of pulse replications [1,6,10,12,18–20,22,23]. We may divide the behaviors of self-replication into five classes.

(i) Kerner *et al.* have studied theoretically self-replication of a motionless pulse in an excitable reaction-diffusion system [23]. In this case, a motionless pulse solution becomes unstable by changing the parameters when the pulse width exceeds a certain critical value and splits symmetrically into two pulses. A similar replication is also found in the Gray-Scott model [20].

(ii) A breathing pulse self-replicates at the instant that its width becomes maximum as in Fig. 3(b). The behavior in Fig. 5(b) may be included in this class. To our knowledge, this type of replication, which causes the SG's in Figs. 6(b) and 8(a), has not been reported.

(iii) A pulse increases its width monotonically and splits into two pulses. This generates SG's in Figs. 8(b), 9(a), and 9(b).

(iv) A propagating pulse produces a daughter pulse at the tail region. This has been found in the Gray-Scott model [1,8] and an exothermic reaction-diffusion model [9].

(v) A pulse becomes very small and then two pulses emerge. This type of self-replication occurs in the BvP model with a hyperbolic tangent nonlinearity as shown in Fig. 7(a) and is an origin of SG in Fig. 1.

At present, theoretical analysis of these self-replications has not been developed. Therefore, we do not exclude the possibility that some of the above classes can be related with each other and be unified eventually.

We have classified five types of self-replication in reaction-diffusion systems. It should be mentioned, however, that even when a self-replication occurs, a spatiotemporal pattern does not always become an SG. A key factor for an SG with $k=2$ is the successive appearance of the same self-replication. This can be understood by looking at Fig. 6(c), where an SG is not generated. One notes that the contour of the pulses at either end gradually changes in time. In order for an SG to be realized, pulses generated by self-replication must obey the same dynamic motion as the mother pulse. This is the reason why one has to tune the parameter to produce an SG.

As was mentioned in the Introduction, a related study has been done by Meinhardt [11]. However, we emphasize that the various beautiful shell patterns obtained by Meinhardt in reaction-diffusion systems are neither perfectly regular nor self-similar. What we have shown in the present paper is that excitable reaction-diffusion systems are capable of generating a regular self-similar spatiotemporal pattern. This implies that, from the viewpoint of pattern formation, some class of solution in reaction-diffusion equations has a relationship, in a pronounced way, with discrete model systems such as cellular automata.

Although we have focused our attention on emergence of the regular fractal structures in the present paper, more complicated spatiotemporal patterns such as those in Figs. 6(a), 6(c), 6(d), and 6(e) should also be analyzed in further details. At present, we do not have any definite conclusion whether these are really chaotic or not. A two-dimensional extension would also be interesting in connection with recent study of spiral breakup and wave instabilities in excitable reaction-diffusion media with fast inhibitor diffusion [24,25]. We shall report on these problems elsewhere in the near future.

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